

Applications of the Square Sieve to a Conjecture of Lang and Trotter for a Pair of Elliptic Curves Over the Rationals



S. Baier and Vijay M. Patankar

To V. Kumar Murty: On the occasion of his sixtieth birthday

Abstract Let E be an elliptic curve over \mathbb{Q} . Let p be a prime of good reduction for E . Then, for a prime $\ell \neq p$, the Frobenius automorphism associated with p (unique up to conjugation) acts on the ℓ -adic Tate module of E . The characteristic polynomial of the Frobenius automorphism has rational integer coefficients and is independent of ℓ . Its splitting field is called the Frobenius field of E at p . Let E_1 and E_2 be two elliptic curves defined over \mathbb{Q} that are non-isogenous over $\overline{\mathbb{Q}}$ and also without complex multiplication over $\overline{\mathbb{Q}}$. In analogy with the well-known Lang-Trotter conjecture for a single elliptic curve, it is natural to consider the asymptotic behaviour of the function that counts the number of primes $p \leq x$ such that the Frobenius fields of E_1 and E_2 at p coincide. In this short note, using Heath-Brown's square sieve, we provide both conditional (upon the Generalized Riemann Hypothesis) and unconditional upper bounds.

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S. Baier

Department of Mathematics, R. K. M. Vivekananda University,
P. O. Belurmah, Howrah 711202, West Bengal, India
e-mail: stephanbaier2017@gmail.com

V. M. Patankar (✉)

School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110067, India
e-mail: vijaypatankar@gmail.com

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1 Introduction and Statement of the Main Theorem

In [6], Lang and Trotter state their conjectures about the distribution of images of Frobenius automorphisms in GL_2 extensions of \mathbb{Q} . These conjectures can be explicitly stated in the context of elliptic curves and their associated Galois representations. A generalization of these conjectures in the setting of a strictly compatible system of λ -adic Galois representations over a number field was developed by V. Kumar Murty in [7].

Let E be an elliptic curve defined over the field of rational numbers with conductor N . Let $G_{\mathbb{Q}}$ denote the absolute Galois group of \mathbb{Q} , i.e. the Galois group of $\overline{\mathbb{Q}}$ over \mathbb{Q} . For a prime ℓ , let ρ_{ℓ} be the associated representation of $G_{\mathbb{Q}}$ into $\mathrm{GL}_2(\mathbb{Z}_{\ell})$ that represents the action of $G_{\mathbb{Q}}$ on the ℓ -adic Tate module attached to E . Then, a prime $p \nmid N\ell$ is unramified in the subfield of $\overline{\mathbb{Q}}$ defined as the fixed field of $\mathrm{Ker}(\rho_{\ell})$. Thus, the Frobenius automorphism attached to p in $G_{\mathbb{Q}}/\mathrm{Ker}(\rho_{\ell})$ is well defined up to conjugation. We denote the associated conjugacy class by σ_p . As a consequence, the characteristic polynomial of $\rho_{\ell}(\sigma_p)$, say, $\phi_p(x) := x^2 - t_p x + p$, is well defined. It is known that the characteristic polynomial has rational integer coefficients, is independent of ℓ , and $t_p = 1 + p - \#E(\mathbb{F}_p)$, where $\#E(\mathbb{F}_p)$ is the number of \mathbb{F}_p -rational points of the reduction of E modulo p . Furthermore, $\phi_p(x)$ has complex conjugate roots of absolute value \sqrt{p} . We then define the Frobenius field associated with E at p as $F(E, p) := \mathbb{Q}\left(\sqrt{t_p^2 - 4p}\right)$. Note that $F(E, p)$ is the splitting field of $\phi_p(x)$.

For non-negative real valued functions f and g , we write $f \ll g$ or $f(x) \ll g(x)$ or $f = O(g)$ or $f(x) = O(g(x))$, if for some positive constant K , $f(x) \leq Kg(x)$ for sufficiently large x . If f and g are not non-negative functions or not real valued functions, then by $f \ll g$ etc., we mean $|f| \ll |g|$. We write, $f(x) \sim g(x)$, if $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$.

We now state the precise forms of the Lang and Trotter conjectures for a single elliptic curve E (with notations as above). The details about the explicit (product) formula for $C(E, t)$ in Conjecture 1 can be found in [6, section 4] – especially [6, page 33] and [6, Theorem 4.2, page 36].

Conjecture 1 (Fixed trace) *Let E be an elliptic curve defined over the rational numbers and without complex multiplication. Let t be an integer. Let*

$$S(E, t; x) := \#\{p \leq x : p \nmid N, t_p = t\}.$$

Then, there exists a constant $C(E, t)$ such that

$$S(E, t; x) \sim C(E, t) \frac{\sqrt{x}}{\log x}$$

as x tends to $+\infty$.

Remark 1 In the above, $C(E, 0) > 0$. On the other hand, when $t \neq 0$ it may happen that $C(E, t) = 0$. This is because the presence of non-trivial torsion points in $E(\mathbb{Q})$

may impose congruence conditions on the traces of the Frobenius automorphism. The details can be found in [6, Remark 1, page 37]. When $C(E, t) = 0$, the conjecture will be interpreted to mean that $S(E, t; x)$ is a bounded function.

Conjecture 2 (Fixed Frobenius field) *Let E be an elliptic curve defined over the rational numbers and without complex multiplication. Let F be an imaginary quadratic field. Let*

$$S(E, F; x) := \#\{p \leq x : p \nmid N, F(E, p) = F\}.$$

Then, there exists a constant $C(E, F) > 0$ such that

$$S(E, F; x) \sim C(E, F) \frac{\sqrt{x}}{\log x}$$

as x tends to $+\infty$.

The details about the above conjecture can be found on page 69, page 109, Theorem 5.1 on page 110, and Theorem 6.3 on page 116 of [6].

Based on heuristic arguments as explained in [6, Remark 2, pages 37–38], Lang and Trotter make the following conjecture regarding coincidence of supersingular primes for a pair of elliptic curves. Let E_1 and E_2 be two elliptic curves over \mathbb{Q} both without complex multiplication and with conductors N_1 and N_2 , respectively. For $p \nmid N_1 N_2$, let $a_p(E_1) := p + 1 - \#E_1(\mathbb{F}_p)$ and $a_p(E_2) := p + 1 - \#E_2(\mathbb{F}_p)$.

Conjecture 3 (Supersingular primes for a pair of elliptic curves) *With notation as above, let*

$$S(E_1, E_2, 0; x) := \#\{p \leq x : p \nmid N_1 N_2, a_p(E_1) = a_p(E_2) = 0\}.$$

Then,

$$S(E_1, E_2, 0; x) = O(\log \log x)$$

if and only if E_1 and E_2 are non-isogenous over $\overline{\mathbb{Q}}$.

Remark 2 Suppose E_1 and E_2 are isogenous over \mathbb{Q} . Then, by Conjecture 1, and the fact that $C(E, 0) > 0$, it follows that $S(E_1, E_2, 0; x) \sim C(E, 0) \frac{\sqrt{x}}{\log x} \neq O(\log \log x)$.

Since the submission of this paper and the previous [5], an explicit version of a more general conjecture has been proposed by Akbary and Park in [1].

Conjecture 4 ([1, Conjecture 1.2] Pair of elliptic curves and pair of fixed traces) *Let E_1 and E_2 be two non-isogenous elliptic curves defined over \mathbb{Q} with conductors N_1 and N_2 , respectively, and both without complex multiplication. For fixed integers t_1 and t_2 , let*

$$S(E_1, E_2, t_1, t_2; x) := \#\{p \leq x : p \nmid N_1 N_2, a_p(E_1) = t_1 \text{ and } a_p(E_2) = t_2\}.$$

Then, there exists a constant $C(E_1, E_2, t_1, t_2) \geq 0$ such that

$$S(E_1, E_2, t_1, t_2; x) \sim C(E_1, E_2, t_1, t_2) \log \log x$$

as $x \rightarrow \infty$.

The conjectural constant $C(E_1, E_2, t_1, t_2)$ is the focus of their paper. Thus, setting $t_1 = t_2 = 0$, one can recover a precise form of Conjecture 3.

Similarly, we would like to study the function that counts the number of primes of good reduction $p \leq x$ such that the corresponding Frobenius fields for E_1 and E_2 are equal, i.e.

$$S(E_1, E_2; x) := \#\{p \leq x : p \nmid N_1 N_2, F(E_1, p) = F(E_2, p)\}. \quad (1)$$

Let us note that if E is an elliptic curve over \mathbb{Q} without complex multiplication, then the set of Frobenius fields $F(E, p)$ as p runs over the primes of good reduction for E is an infinite set. This follows from a series of exercises in Serre's book [10, Chapter IV, pages 13–14].

In this context, we quote the last sentence on page 38 of [6, Remark 2]: “*Of course a similar conjecture can be made about primes whose Frobenius elements for two given curves both generate the same quadratic field.*” This statement is not very explicit but it seems to indicate a plausible conjectural answer to the asymptotic behaviour of $S(E_1, E_2; x)$.

Conjecture 5¹ ([5, Conjecture 1] Pair of elliptic curves and equal Frobenius fields) *Let E_1 and E_2 be two elliptic curves over the rationals, and both without complex multiplication over $\overline{\mathbb{Q}}$. Then, E_1 is not isogenous to E_2 over $\overline{\mathbb{Q}}$ if and only if*

$$S(E_1, E_2; x) = O(\log \log x).$$

Remark 3 Suppose E_1 and E_2 are isogenous over $\overline{\mathbb{Q}}$, hence over some number field L . By extending L if necessary, we can assume that L is Galois over \mathbb{Q} . Then, for any prime $p \nmid N_1 N_2$ and that splits in L , $a_p(E_1) = a_p(E_2)$. Thus, from the Chebotarev density theorem, it follows that $S(E_1, E_2; x)$ grows at least as much as $\frac{1}{[L:\mathbb{Q}]} \frac{x}{\log x}$. Hence, $S(E_1, E_2; x) \neq O(\log \log x)$.

Using techniques from ℓ -adic representations, the following result was proved in [5].

Theorem 1 [5, Theorem 3] *Let E_1 and E_2 be two elliptic curves over a number field K . Let Σ_r be a finite subset of the set Σ_K of finite places of K containing the places of bad reduction of E_1 and E_2 . Assume that at least one of the elliptic curves is without complex multiplication. Let*

$$S(E_1, E_2) := \{v \in \Sigma_K \setminus \Sigma_r : F(E_1, v) = F(E_2, v)\}.$$

¹Unfortunately and inadvertently, this conjecture was wrongly stated in [5, Conjecture 1].

Then, E_1 and E_2 are isogenous over a finite extension of K if and only if $S(E_1, E_2)$ has positive upper density.

In the above, the upper density of a subset of primes S is defined as

$$ud(S) := \limsup_{x \rightarrow \infty} \frac{\#\{v \in \Sigma_K \setminus \Sigma_r : Nv \leq x \text{ and } v \in S\}}{\#\{v \in \Sigma_K : Nv \leq x\}},$$

where Nv denotes the cardinality of the residue field of K at v .

Remark 4 If p is a common supersingular prime, then $F(E_1, p) = F(E_2, p) = \mathbb{Q}(\sqrt{-p})$. Thus, Conjectures 3 and 5 are entangled.

In this note, we establish non-trivial upper bounds on $S(E_1, E_2; x)$, both conditional and unconditional. Our results are stated below.

Assuming the Generalized Riemann Hypothesis (GRH) for the Dedekind zeta functions of number fields, we are able to prove the following assertion.

Theorem 2 *Let E_1 and E_2 be two elliptic curves defined over \mathbb{Q} . Suppose E_1 and E_2 are non-isogenous over $\overline{\mathbb{Q}}$, and also both without complex multiplication over $\overline{\mathbb{Q}}$. Then, under GRH, we have*

$$S(E_1, E_2; x) \ll x^{29/30} (\log x)^{1/15},$$

where the implied constant depends only on E_1 and E_2 .

Without assuming the GRH, we prove the following result.

Theorem 3 *Let E_1 and E_2 be two elliptic curves defined over \mathbb{Q} . Suppose E_1 and E_2 are non-isogenous over $\overline{\mathbb{Q}}$, and also both without complex multiplication over $\overline{\mathbb{Q}}$. Then,*

$$S(E_1, E_2; x) \ll \frac{x(\log \log x)^{22/21}}{(\log x)^{43/42}}.$$

Remark 5 Note that Theorem 3 implies Theorem 1 by the prime number theorem under the additional assumption that both the elliptic curves are without complex multiplication.

Remark 6 It is mentioned in the last paragraph of [2, Section 6, pages 1174–1175] that as an application of Theorem 10 of [9] one can obtain the following estimate under the GRH.

$$S(E_1, E_2; x) = O_{E_1, E_2}(x^{\frac{11}{12}}),$$

where the O -constant depends in an unspecified way on E_1 and E_2 . This is a better bound than the one obtained in Theorem 2, but here the implied O -constant depends in an unspecified way on E_1 and E_2 , whereas in Theorems 2 and 3, the implied

constants are effectively computable. Moreover, our method is different and a rather easy application of the square sieve and the Chebotarev density theorem.

2 Square-Sieve Approach

Our technique closely follows that of [2], where the authors apply Heath-Brown's square sieve to bound the function that counts the number of primes $p \leq x$ such that the associated Frobenius field at p equals the given imaginary quadratic field.

Clearly, $S(E_1, E_2; x)$, as defined in (1), equals the number of primes $p \leq x$ with $p \nmid N$ such that

$$\begin{aligned} 4p - a_p^2 &= Dm^2 \\ 4p - b_p^2 &= Dn^2 \end{aligned} \tag{2}$$

for some squarefree D and natural numbers m and n , where $a_p := a_p(E_1)$ and $b_p := a_p(E_2)$. For a given p , it is easy to see that the above system is satisfied for a squarefree D and m, n natural numbers if and only if $(4p - a_p^2)(4p - b_p^2)$ is a square. To see this, we note that if

$$\begin{aligned} 4p - a_p^2 &= D_1 m^2 \\ 4p - b_p^2 &= D_2 n^2 \end{aligned} \tag{3}$$

with D_1 and D_2 squarefree, then

$$(4p - a_p^2)(4p - b_p^2) = D_1 D_2 m^2 n^2$$

if and only if $D_1 = D_2$. Hence,

$$S(E_1, E_2; x) = \# \left\{ p \leq x : p \nmid N, (4p - a_p^2)(4p - b_p^2) = \square \right\},$$

where $N = N_1 N_2$ is the product of the conductors N_1 and N_2 of E_1 and E_2 , respectively. We detect squares in the multiset

$$\mathcal{A} := \{(4p - a_p^2)(4p - b_p^2) : p \leq x, p \nmid N\} \tag{4}$$

by using a version of Heath-Brown's square sieve. The version 1 below, which is a consequence of Theorem 1 in [3], is best suited for obtaining a conditional bound under GRH. The version 2, as stated in Sect. 7, is better suited for an unconditional version (without assuming GRH) of the theorem.

Lemma 4 (square sieve - version 1) *Let \mathcal{A} be a multiset of positive integers, and let \mathcal{P} be a set of P distinct primes. Set*

$$S(\mathcal{A}) := \#\{\alpha \in \mathcal{A} : \alpha \text{ is a square}\}$$

and suppose that

$$\max_{n \in \mathcal{A}} n \leq e^P. \quad (5)$$

Then,

$$\#S(\mathcal{A}) \ll \frac{\#\mathcal{A}}{P} + \frac{1}{P^2} \cdot \sum_{\substack{q_1, q_2 \in \mathcal{P} \\ q_1 \neq q_2}} \left| \sum_{n \in \mathcal{A}} \left(\frac{n}{q_1 q_2} \right) \right|, \quad (6)$$

where $\left(\frac{n}{q_1 q_2} \right)$ is the Jacobi symbol.

We now apply this lemma with \mathcal{A} as in (4),

$$\mathcal{P} := \{q \in \Sigma_{\mathbb{Q}} : z/2 < q \leq z\} \quad \text{and} \quad P := \#\mathcal{P}, \quad (7)$$

where $\Sigma_{\mathbb{Q}}$ denotes the set of primes of \mathbb{Q} and z is a positive parameter satisfying

$$e^P = e^{\pi(z) - \pi(z/2)} > x \quad (8)$$

which will be suitably chosen later. It follows that

$$S(E_1, E_2; x) \ll 1 + \frac{1}{P} \cdot (\pi(x) - \omega(N)) + \frac{1}{P^2} \cdot \sum_{\substack{q_1, q_2 \in \mathcal{P} \\ q_1 \neq q_2}} \left| \sum_{\substack{p \leq x \\ p \nmid N}} \left(\frac{(4p - a_p^2)(4p - b_p^2)}{q_1 q_2} \right) \right|, \quad (9)$$

where $\omega(N)$ is the numbers of distinct prime factors of N . Moreover, by the prime number theorem,

$$P \sim \frac{z}{2 \log z} \quad \text{and} \quad \pi(x) - \omega(N) \sim \frac{x}{\log x}. \quad (10)$$

Thus, to ensure that (8) is satisfied for large enough x , it suffices that

$$z > (\log x)^{1+\varepsilon}, \quad (11)$$

which we assume from now on.

We now estimate the character sum in (9). We break the summation into residue classes as follows.

$$\sum_{\substack{p \leq x \\ p \nmid q_1 q_2 N}} \left(\frac{(4p - a_p^2)(4p - b_p^2)}{q_1 q_2} \right) = \sum_{\substack{d=1 \\ (d, q_1 q_2)=1}}^{q_1 q_2} \sum_{s=1}^{q_1 q_2} \sum_{t=1}^{q_1 q_2} \left(\frac{(4d - s^2)(4d - t^2)}{q_1 q_2} \right) \cdot \pi(x; d, s, t), \quad (12)$$

where

$$\pi(x; d, s, t) := \#\{p \leq x : p \nmid N, p \equiv d \pmod{q_1 q_2}, \\ a_p \equiv s \pmod{q_1 q_2}, b_p \equiv t \pmod{q_1 q_2}\}. \quad (13)$$

The task is now to evaluate $\pi(x; d, s, t)$ asymptotically.

We first recall some group theoretical results from [2]. We will then apply versions of the Chebotarev density theorem (under GRH or without) and obtain our results.

3 Application of Chebotarev Density Theorem Under the GRH

For $i = 1, 2$ let $E_i[k]$ be the group of k -division points of E_i and $F_k^i := \mathbb{Q}(E_i[k])$ be the field obtained by adjoining to \mathbb{Q} the x and the y coordinates of k -division points. Let

$$G_k^i := \text{Gal}(F_k^i/\mathbb{Q}). \quad (14)$$

The following facts were used in [2] and are of importance here as well. The action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the k -division points on E_i gives rise to the natural Galois representations

$$\phi_k^i : G_k^i \rightarrow \text{GL}_2(\mathbb{Z}/k\mathbb{Z}) \quad (15)$$

which are injective for every $k \in \mathbb{N}$. By [8, Theorem 2 on page 37 and Theorem 3 on page 42], it follows that for elliptic curves E_i without complex multiplication, there exist constants $C(E_i)$ depending on E_i such that ϕ_k^i are surjective whenever $(k, C(E_i)) = 1$.

In our applications, $k = q_1 q_2$, and so the condition $(q_1 q_2, C(E_i)) = 1$ will be satisfied for $i = 1, 2$ if x is large enough since we have assumed that $z/2 < q_1, q_2 \leq z$ and $z > (\log x)^{1+\varepsilon}$. Hence, we have

$$G_{q_1 q_2}^i \cong \text{GL}_2(\mathbb{Z}/q_1 q_2 \mathbb{Z}), \quad \text{if } x \text{ is large enough.}$$

Further, ϕ_k^i has the properties that

$$\mathrm{tr}(\phi_k^1(\sigma_p^1)) = a_p, \quad \mathrm{tr}(\phi_k^2(\sigma_p^2)) = b_p, \quad \det(\phi_k^1(\sigma_p^1)) = p = \det(\phi_k^2(\sigma_p^2)),$$

where σ_p^i is the Frobenius conjugacy class associated with p in $\mathrm{Gal}(F_k^i/\mathbb{Q})$. Equipped with the above, we are now ready for an application of the Chebotarev density theorem under GRH (see [2, Theorem 2.3]).

Theorem 5 (Chebotarev density theorem under GRH) *Let L be a finite Galois extension of \mathbb{Q} with Galois group G . Let n_L be the degree of L over \mathbb{Q} and d_L its discriminant. Let C be a union of conjugacy classes of G . Define*

$$\pi_C(x, L, \mathbb{Q}) := \#\{p \leq x : p \text{ unramified in } L/\mathbb{Q}, \sigma_p \subseteq C\},$$

where σ_p is the Frobenius conjugacy class associated with p in the extension L/\mathbb{Q} . Then, under the GRH for Dedekind zeta functions, we have

$$\pi_C(x, L, \mathbb{Q}) = \frac{\#C}{\#G} \cdot \mathrm{li}(x) + O\left(\#C \cdot x^{1/2} \left(\log x + \frac{\log d_L}{n_L}\right)\right),$$

where the implicit constants are absolute.

We wish to apply the above theorem for $L := F_{q_1 q_2}^1 F_{q_1 q_2}^2$. Let $H_{q_1 q_2}$ be the Galois group $\mathrm{Gal}(L/\mathbb{Q})$. We now use Serre's results on the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ under the product of ℓ -adic representations attached to two non-isogenous elliptic curves (both without complex multiplication). In fact, [8, Corollary 2, page 324] implies that

$$H_{q_1 q_2} = \left\{ (g_1, g_2) \in G_{q_1 q_2}^1 \times G_{q_1 q_2}^2 : \det \phi_{q_1 q_2}^1(g_1) = \det \phi_{q_1 q_2}^2(g_2) \right\},$$

with $\phi_{q_1 q_2}^i$ ($i = 1, 2$) as in (15). Let C be the following conjugacy class in $G = H_{q_1 q_2}$.

$$C := C_{q_1 q_2}(s, t, d) := \left\{ (g_1, g_2) \in G_{q_1 q_2}^1 \times G_{q_1 q_2}^2 : \det \phi_{q_1 q_2}^1(g_1) = d = \det \phi_{q_1 q_2}^2(g_2), \right. \\ \left. \mathrm{tr} \phi_{q_1 q_2}^1(g_1) = s, \mathrm{tr} \phi_{q_1 q_2}^2(g_2) = t \right\}.$$

Applying Theorem 5, we get

$$\pi(x; d, s, t) = \frac{\#C_{q_1 q_2}(s, t, d)}{\#H_{q_1 q_2}} \cdot \mathrm{li}(x) + O\left(\#C_{q_1 q_2}(s, t, d) x^{1/2} \left(\log x + \frac{\log d_L}{n_L}\right)\right) \quad (16)$$

under GRH.

Combining (12) and (16), and taking into account that

$$\sum_{\substack{d=1 \\ (d, q_1 q_2)=1}}^{q_1 q_2} \sum_{s=1}^{q_1 q_2} \sum_{t=1}^{q_1 q_2} \#C_{q_1 q_2}(s, t, d) = \#H_{q_1 q_2}, \quad (17)$$

we have

$$\begin{aligned} & \sum_{\substack{p \leq x \\ p \nmid q_1 q_2 N}} \left(\frac{(4d - a_p^2)(4d - b_p^2)}{q_1 q_2} \right) \\ = & \text{li}(x) \cdot \sum_{\substack{d=1 \\ (d, q_1 q_2)=1}}^{q_1 q_2} \sum_{s=1}^{q_1 q_2} \sum_{t=1}^{q_1 q_2} \left(\frac{(4d - s^2)(4d - t^2)}{q_1 q_2} \right) \cdot \frac{\#C_{q_1 q_2}(s, t, d)}{\#H_{q_1 q_2}} + \\ & O \left(\#H_{q_1 q_2} x^{1/2} \left(\log x + \frac{\log d_L}{n_L} \right) \right). \end{aligned} \quad (18)$$

4 Counting

We now compute $\#C_{q_1 q_2}(s, t, d)$ and $\#H_{q_1 q_2}$ that appear on the right-hand side of Eq.(18).

Since, for q_1, q_2 large enough,

$$\phi_{q_1 q_2}^i : F_{q_1 q_2}^i \rightarrow \text{GL}_2(\mathbb{Z}/q_1 q_2 \mathbb{Z}), \quad i = 1, 2$$

are isomorphisms, we have

$$\#C_{q_1 q_2}(s, t, d) = \#\{(A_1, A_2) \in \text{GL}_2(\mathbb{Z}/q_1 q_2 \mathbb{Z})^2 : \det(A_1) = d = \det(A_2), \text{tr}(A_1) = s, \text{tr}(A_2) = t\} \quad (19)$$

and

$$\#H_{q_1 q_2} = \#\{(A_1, A_2) \in \text{GL}_2(\mathbb{Z}/q_1 q_2 \mathbb{Z})^2 : \det(A_1) = \det(A_2)\}. \quad (20)$$

We now use Corollary 2.8 from [2] which states the following.

Corollary 1 *Let q_1 and q_2 be two distinct odd primes, and $d, t \in \mathbb{Z}/q_1q_2\mathbb{Z}$ be fixed with $(d, q_1q_2) = 1$. Then,*

$$\begin{aligned} & \#\{g \in \mathrm{GL}_2(\mathbb{Z}/q_1q_2\mathbb{Z}) : \det(g) = d, \mathrm{tr}(g) = t\} \\ &= q_1q_2 \left(q_1 + \left(\frac{t^2 - 4d}{q_1} \right) \right) \left(q_2 + \left(\frac{t^2 - 4d}{q_2} \right) \right). \end{aligned}$$

Applying Corollary 1, we deduce that

$$\begin{aligned} \#C_{q_1q_2}(s, t, d) &= q_1^2q_2^2 \left(q_1 + \left(\frac{s^2 - 4d}{q_1} \right) \right) \left(q_2 + \left(\frac{s^2 - 4d}{q_2} \right) \right) \times \\ & \quad \left(q_1 + \left(\frac{t^2 - 4d}{q_1} \right) \right) \left(q_2 + \left(\frac{t^2 - 4d}{q_2} \right) \right). \end{aligned}$$

Since $\{(A_1, A_2) \in \mathrm{GL}_2(\mathbb{Z}/q_1q_2\mathbb{Z})^2 : \det(A_1) = \det(A_2)\}$ is the kernel of the epimorphism

$$\sigma : \mathrm{GL}_2(\mathbb{Z}/q_1q_2\mathbb{Z})^2 \rightarrow (\mathbb{Z}/q_1q_2\mathbb{Z})^*$$

defined by

$$\sigma(A_1, A_2) = \det(A_1) \cdot \det(A_2)^{-1},$$

it follows that

$$\begin{aligned} \#H_{q_1q_2} &= \frac{\#\mathrm{GL}_2(\mathbb{Z}/q_1q_2\mathbb{Z})^2}{\#(\mathbb{Z}/q_1q_2\mathbb{Z})^*} = \frac{q_1^2(q_1 - 1)^2(q_1^2 - 1)^2q_2^2(q_2 - 1)^2(q_2^2 - 1)^2}{(q_1 - 1)(q_2 - 1)} \\ &= q_1^2(q_1 - 1)(q_1^2 - 1)^2q_2^2(q_2 - 1)(q_2^2 - 1)^2. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\#C_{q_1q_2}(s, t, d)}{\#H_{q_1q_2}} &= \frac{1}{(q_1 - 1)(q_1^2 - 1)^2(q_2 - 1)(q_2^2 - 1)^2} \times \\ & \quad \left(q_1 + \left(\frac{s^2 - 4d}{q_1} \right) \right) \left(q_2 + \left(\frac{s^2 - 4d}{q_2} \right) \right) \times \\ & \quad \left(q_1 + \left(\frac{t^2 - 4d}{q_1} \right) \right) \left(q_2 + \left(\frac{t^2 - 4d}{q_2} \right) \right) \\ &= \frac{q_1^2q_2^2}{(q_1 - 1)(q_1^2 - 1)^2(q_2 - 1)(q_2^2 - 1)^2} + O\left(\frac{1}{z^7}\right) \end{aligned} \tag{21}$$

if $z/2 < q_1, q_2 \leq z$. This allows us to express the main term on the right-hand side of (18) explicitly. For the estimations of the O -term in (18) and an O -term occurring

later in this paper, we prove the following estimates for n_L , the degree of L/\mathbb{Q} , and d_L , the discriminant of L/\mathbb{Q} .

Lemma 6 *Suppose that $z/2 < q_1, q_2 \leq z$. Then, we have*

$$n_L \leq z^{14}, \quad \frac{\log |d_L|}{n_L} \ll \log z,$$

$$|d_L|^{1/n_L} \ll z^{16} \quad \text{and} \quad \log |d_L| \ll z^{14} \log z.$$

Proof Clearly,

$$n_L = \#H_{q_1 q_2} = q_1^2(q_1 - 1)(q_1^2 - 1)^2 q_2^2(q_2 - 1)(q_2^2 - 1)^2 \ll z^{14}. \quad (22)$$

By a lemma of Hensel (see Lemma 2.6 in [2]), we have

$$\log |d_L| \leq n_L \log n_L + (n_L - 1) \sum_{p \text{ ramified in } L} \log p$$

which implies that

$$\frac{\log |d_L|}{n_L} \leq \log n_L + \log(q_1 q_2 N) \leq 8 \log(q_1 q_2) + \log(N), \quad (23)$$

where we use the fact that if p ramifies in L , then $p|(q_1 q_2 N)$, which follows from the Néron-Ogg-Shafarevich criterion. From (23), we deduce that

$$|d_L|^{1/n_L} = \exp\left(\frac{\log |d_L|}{n_L}\right) \leq N(q_1 q_2)^8 \ll z^{16},$$

and from (22) and (23), we deduce that

$$\log |d_L| \leq n_L(\log n_L + \log(q_1 q_2 N)) \ll z^{14} \log z.$$

□

Plugging (21) into (18) and using Lemma 6 gives

$$\begin{aligned}
& \sum_{\substack{p \leq x \\ p \nmid q_1 q_2 N}} \left(\frac{(4d - a_p^2)(4d - b_p^2)}{q_1 q_2} \right) \\
&= \text{li}(x) \cdot \frac{q_1^2 q_2^2}{(q_1 - 1)(q_1^2 - 1)^2 (q_2 - 1)(q_2^2 - 1)^2} \cdot \sum_{\substack{d=1 \\ (d, q_1 q_2)=1}}^{q_1 q_2} \sum_{s=1}^{q_1 q_2} \sum_{t=1}^{q_1 q_2} \left(\frac{(4d - s^2)(4d - t^2)}{q_1 q_2} \right) + \\
& O\left(\frac{\text{li}(x)}{z} + x^{1/2} z^{14} \log(xz) \right).
\end{aligned} \tag{24}$$

5 Evaluation of Character Sums

Next, we evaluate the character sums above. We have

$$\begin{aligned}
& \sum_{\substack{d=1 \\ (d, q_1 q_2)=1}}^{q_1 q_2} \sum_{s=1}^{q_1 q_2} \sum_{t=1}^{q_1 q_2} \left(\frac{(4d - s^2)(4d - t^2)}{q_1 q_2} \right) = \sum_{\substack{d=1 \\ (d, q_1 q_2)=1}}^{q_1 q_2} \left(\sum_{u=1}^{q_1 q_2} \left(\frac{4d - u^2}{q_1 q_2} \right) \right)^2 \\
&= \sum_{\substack{d=1 \\ (d, q_1)=1}}^{q_1} \left(\sum_{v=1}^{q_1} \left(\frac{4d - v^2}{q_1} \right) \right)^2 \cdot \sum_{\substack{d=1 \\ (d, q_2)=1}}^{q_2} \left(\sum_{w=1}^{q_2} \left(\frac{4d - w^2}{q_2} \right) \right)^2.
\end{aligned} \tag{25}$$

For an odd prime q with $(d, q) = 1$, we may write

$$\begin{aligned}
& \sum_{x \bmod q} \left(\frac{4d - x^2}{q} \right) = \frac{1}{2} \cdot \sum_{\substack{y \bmod q \\ y \neq 0 \bmod q}} \left(1 + \left(\frac{y}{q} \right) \right) \left(\frac{4d - y}{q} \right) + \left(\frac{4d}{q} \right) \\
&= \frac{1}{2} \cdot \sum_{\substack{y \bmod q \\ y \neq 0 \bmod q}} \left(\frac{4d - y}{q} \right) + \frac{1}{2} \cdot \sum_{\substack{y \bmod q \\ y \neq 0 \bmod q}} \left(\frac{y}{q} \right) \left(\frac{4d - y}{q} \right) + \left(\frac{4d}{q} \right) \\
&= \frac{1}{2} \cdot \sum_{y \bmod q} \left(\frac{4d - y}{q} \right) + \frac{1}{2} \cdot \sum_{y \bmod q} \left(\frac{y}{q} \right) \left(\frac{4d - y}{q} \right) + \frac{1}{2} \cdot \left(\frac{4d}{q} \right).
\end{aligned}$$

Using the orthogonality relations for Dirichlet characters, the first sum in the last line equals 0, and hence, the above simplifies into

$$\sum_{x \bmod q} \left(\frac{4d - x^2}{q} \right) = \frac{1}{2} \cdot \sum_{y \bmod q} \left(\frac{y}{q} \right) \left(\frac{4d - y}{q} \right) + \frac{1}{2} \cdot \left(\frac{4d}{q} \right). \tag{26}$$

Now we reduce the right-hand side to Jacobi sums by writing

$$\begin{aligned}
& \frac{1}{2} \cdot \sum_{y \bmod q} \left(\frac{y}{q}\right) \left(\frac{4d-y}{q}\right) + \frac{1}{2} \cdot \left(\frac{4d}{q}\right) \\
&= \frac{1}{2} \cdot \left(\frac{4d}{q}\right)^2 \cdot \sum_{y \bmod q} \left(\frac{y\overline{4d}}{q}\right) \left(\frac{1-y\overline{4d}}{q}\right) + \frac{1}{2} \cdot \left(\frac{4d}{q}\right) \\
&= \frac{1}{2} \cdot \sum_{z \bmod q} \left(\frac{z}{q}\right) \left(\frac{1-z}{q}\right) + \frac{1}{2} \cdot \left(\frac{d}{q}\right).
\end{aligned} \tag{27}$$

The last line is evaluated to be

$$\frac{1}{2} \cdot \sum_{z \bmod q} \left(\frac{z}{q}\right) \left(\frac{1-z}{q}\right) + \frac{1}{2} \cdot \left(\frac{d}{q}\right) = \frac{1}{2} \cdot \left(-\left(\frac{-1}{q}\right) + \left(\frac{d}{q}\right)\right) \in \{-1, 0, 1\}, \tag{28}$$

where we apply the following well-known evaluation of Jacobi sums defined as

$$J(\chi, \lambda) := \sum_{a+b=1} \chi(a)\lambda(b),$$

for the special case $\lambda = \chi^{-1} = (\cdot/q)$.

Lemma 7 *We have*

$$J(\chi, \chi^{-1}) = -\chi(-1).$$

Proof This is [4, page 93, Theorem 1(c)].

□

Combining (26), (27) and (28), we see that

$$\sum_{x \bmod q} \left(\frac{4d-x^2}{q}\right) \in \{-1, 0, 1\}.$$

Plugging this into (25), we deduce that

$$\sum_{\substack{d=1 \\ (d, q_1 q_2)=1}}^{q_1 q_2} \sum_{s=1}^{q_1 q_2} \sum_{t=1}^{q_1 q_2} \left(\frac{(4d-s^2)(4d-t^2)}{q_1 q_2}\right) \leq (q_1-1)(q_2-1). \tag{29}$$

6 Proof of Theorem 2

Plugging (29) into (18), and using (21), we get

$$\begin{aligned} & \sum_{\substack{p \leq x \\ p \nmid q_1 q_2 N}} \left(\frac{(4d - a_p^2)(4d - b_p^2)}{q_1 q_2} \right) \\ & \leq \text{li}(x) \cdot \frac{q_1^2 q_2^2}{(q_1^2 - 1)^2 (q_2^2 - 1)^2} + O\left(\frac{\text{li}(x)}{z} + x^{1/2} z^{14} \log(xz)\right) \\ & = O\left(\frac{\text{li}(x)}{z} + x^{1/2} z^{14} \log(xz)\right) \end{aligned}$$

if $z/2 < q_1, q_2 \leq z$. Combining this with (9) and (10), we deduce that

$$S(E_1, E_2; x) \ll \frac{x/\log x}{z/\log z} + x^{1/2} z^{14} \log(xz).$$

Choosing $z := x^{1/30} (\log x)^{-1/15}$, we obtain Theorem 2.

7 Proof of Theorem 3

In the following, we modify the method in order to establish an unconditional upper bound for $S(E_1, E_2; x)$. To this end, we shall need the following second version of the square sieve and an unconditional effective version of the Chebotarev density theorem stated below.

Lemma 8 (Square sieve, version 2) *Let \mathcal{A} be a multiset of positive integers, and let \mathcal{P} be a set of P distinct primes. Set*

$$S(\mathcal{A}) := \#\{\alpha \in \mathcal{A} : \alpha \text{ is a square}\}.$$

Then

$$S(\mathcal{A}) \leq \frac{\#\mathcal{A}}{P} + \max_{\substack{q_1, q_2 \in \mathcal{P} \\ q_1 \neq q_2}} \left| \sum_{\alpha \in \mathcal{A}} \left(\frac{\alpha}{q_1 q_2} \right) \right| + \frac{2}{P} \cdot \sum_{\alpha \in \mathcal{A}} \sum_{\substack{q \in \mathcal{P} \\ q|\alpha}} 1 + \frac{1}{P^2} \cdot \sum_{\alpha \in \mathcal{A}} \left(\sum_{\substack{q \in \mathcal{P} \\ q|\alpha}} 1 \right)^2. \quad (30)$$

Proof This is Theorem 2.1. in [2] and originated in [3]. \square

Theorem 6 (Chebotarev density theorem, unconditional) *Let the conditions of Theorem 5 be kept. There exist positive constants A , b and b' such that, if*

$$\log x \geq bn_L(\log |d_L|)^2, \quad (31)$$

then

$$\begin{aligned} \pi_C(x, L : \mathbb{Q}) = & \frac{\#C}{\#G} \cdot \text{li}(x) + O\left(\frac{\#C}{\#G} \cdot \text{li}\left(x \cdot \exp\left(-b' \cdot \frac{\log x}{\max\{|d_L|^{1/n_L}, \log |d_L|\}}\right)\right)\right) + \\ & O\left(\left(\#\tilde{C}\right)_x \exp\left(-A\sqrt{\frac{\log x}{n_L}}\right)\right), \end{aligned} \quad (32)$$

where \tilde{C} is the set of conjugacy classes whose union is C .

Proof This is Theorem 2.4 in [2]. \square

Lemma 8 should be compared to Lemma 4. We note that in Lemma 8, the condition (5) is omitted, which shall turn out to be essential for us in order to obtain an unconditional bound. The costs of omitting this condition are two extra terms on the right-hand side of (30), as compared to (6). We shall be able to estimate these terms quite easily using some results in [2].

Applying Lemma 8 in our situation, we now get

$$\begin{aligned} S(E_1, E_2; x) \ll & 1 + \frac{1}{P} \cdot (\pi(x) - \omega(N)) + \\ & \frac{1}{P^2} \cdot \sum_{\substack{q_1, q_2 \in \mathcal{P} \\ q_1 \neq q_2}} \left| \sum_{\substack{p \leq x \\ p \nmid q_1 q_2 N}} \left(\frac{(4p - a_p^2)(4p - b_p^2)}{q_1 q_2} \right) \right| + \\ & \frac{2}{P} \cdot \sum_{p \leq x} \sum_{\substack{q \in \mathcal{P} \\ q \mid (4p - a_p^2)(4p - b_p^2)}} 1 + \frac{1}{P^2} \cdot \sum_{p \leq x} \left(\sum_{\substack{q \in \mathcal{P} \\ q \mid (4p - a_p^2)(4p - b_p^2)}} 1 \right)^2 \end{aligned}$$

with P as defined in (7). To estimate the last two terms on the right-hand side of (33), we write

$$\sum_{p \leq x} \sum_{\substack{q \in \mathcal{P} \\ q | (4p - a_p^2)(4p - b_p^2)}} 1 \leq \sum_{p \leq x} \sum_{\substack{q \in \mathcal{P} \\ q | (4p - a_p^2)}} 1 + \sum_{p \leq x} \sum_{\substack{q \in \mathcal{P} \\ q | (4p - b_p^2)}} 1$$

and

$$\sum_{p \leq x} \left(\sum_{\substack{q \in \mathcal{P} \\ q | (4p - a_p^2)(4p - b_p^2)}} 1 \right)^2 \leq 2 \sum_{p \leq x} \left(\sum_{\substack{q \in \mathcal{P} \\ q | (4p - a_p^2)}} 1 \right)^2 + 2 \sum_{p \leq x} \left(\sum_{\substack{q \in \mathcal{P} \\ q | (4p - b_p^2)}} 1 \right)^2.$$

The bounds (25) and (26) from [2] imply that

$$\sum_{p \leq x} \sum_{\substack{q \in \mathcal{P} \\ q | D(4p - a_p^2)}} 1 + \sum_{p \leq x} \sum_{\substack{q \in \mathcal{P} \\ q | D(4p - b_p^2)}} 1 \ll \frac{x}{\log x}$$

and

$$2 \sum_{p \leq x} \left(\sum_{\substack{q \in \mathcal{P} \\ q | D(4p - a_p^2)}} 1 \right)^2 + 2 \sum_{p \leq x} \left(\sum_{\substack{q \in \mathcal{P} \\ q | D(4p - b_p^2)}} 1 \right)^2 \ll \frac{x}{\log x}$$

for any non-square positive integer D , provided that $z \leq x^{1/2}$. In particular, this holds for $D = 2$. Since all primes q in the above two bounds are odd if $z > 2$, these bounds remain true if we set $D = 1$ in this case. Using (10) in addition, it follows that

$$S(E_1, E_2; x) \ll \frac{x/\log x}{z/\log z} + \frac{1}{P^2} \cdot \sum_{\substack{q_1, q_2 \in \mathcal{P} \\ q_1 \neq q_2}} \left| \sum_{\substack{p \leq x \\ p \nmid q_1 q_2 N}} \left(\frac{(4p - a_p^2)(4p - b_p^2)}{q_1 q_2} \right) \right| \quad (33)$$

if $z > 2$.

Again we shall choose z depending on x so that z exceeds every given real number if x is large enough and hence (14) holds for $z/2 < q_1, q_2 \leq z$. In the following, we assume that we are in this situation. The second term on the right-hand side of (33) is now treated as in the previous sections, but here we apply Theorem 6 in place of Theorem 5. Using the estimates in Lemma 6, the Eq. (17) and $\#\tilde{C} \leq \#C$, we arrive at

$$S(E_1, E_2; x) \ll \frac{x/\log x}{z/\log z} + \frac{x}{\log x} \cdot \exp\left(-c_1 \cdot \frac{\log x}{z^{16}}\right) + z^{14} x \exp\left(-A\sqrt{\frac{\log x}{z^{14}}}\right)$$

for some constants $c_1 > 0$ and $A > 0$, provided that

$$\log x \geq c_2 z^{42} (\log z)^2 \quad (34)$$

for some constant $c_2 > 0$. This condition comes from condition (31) and Lemma 6. Choosing

$$z := c_3 (\log x)^{1/42} (\log \log x)^{-1/21} \quad (35)$$

for some constant $c_3 > 0$ which is small enough so that (35) is consistent with (34), we obtain Theorem 3.

Remark 7 We note that the choice of z in (35) would contradict condition (5) if we had used Lemma 4 instead of Lemma 8. This explains why we work with version 2 of the square sieve in this situation.

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